

No partial erasure of quantum information

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In complete erasure any arbitrary pure quantum state is transformed to a fixed pure state by irreversible operation. Here we ask if the process of partial erasure of quantum information is possible by general quantum operations, where partial erasure refers to reducing the dimension of the parameter space that specifies the quantum state. Here we prove that quantum information stored in qubits and qudits cannot be partially erased, even by irreversible operations. The no-flipping theorem, which rules out the existence of a universal NOT gate for an arbitrary qubit, emerges as a corollary to this theorem. The ‘no partial erasure’ theorem is shown to apply to spin and bosonic coherent states, with the latter result showing that the ‘no partial erasure’ theorem applies to continuous variable quantum information schemes as well. The no partial erasure theorem suggests an integrity principle that quantum information is indivisible.

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I. INTRODUCTION

Classical information can be stored in distinct macroscopic states of a physical system and processed according to classical laws of physics. That ‘information is physical’ is exemplified by the fact that erasure of classical information is an irreversible operation with a cost of $kT \log 2$ of energy per bit operating at a temperature T [1], which is a fundamental source of heat for standard computation [2]. This is the Landauer erasure principle. In quantum information processing, a qubit cannot be erased by a unitary transformation (see Appendix A) and is subject to Landauer’s principle. In recent years considerable effort has been directed toward investigating possible and impossible operations in quantum information theory.

Impossible operations are stated as no-go theorems, which establish limits to quantum information capabilities and also provide intuition to enable further advances in the field. For example the no-cloning theorem [3, 4, 5] underscored the need for quantum error correction to ensure that quantum information processing is possible in faulty systems despite the impossibility of a quantum FANOUT operation. Other examples of important no-go theorems are the no-deletion theorem [6, 7], which proves the impossibility of perfectly deleting one state from two identical states, the no-flipping theorem [8], which establishes the impossibility of designing a universal NOT gate for arbitrary qubit input states, and the impossibility of universal Hadamard and CNOT gates for arbitrary qubit

input states [9]. The strong no-cloning theorem states that the creation of a copy of a quantum state requires full information about the quantum state [10]; together with the no-deletion theorem, these establish permanence of quantum information. A profound consequence of the no-cloning and no-deleting theorems suggest a fundamental principle of conservation of quantum information [11].

Here we establish a new and powerful no-go theorem of quantum information, which suggests both a limitation and protection of quantum information. Our theorem shows that it is impossible to erase quantum information, even partially and even by using irreversible means, where partial erasure corresponds to a reduction of the parameter space dimension for the quantum state that holds the quantum information, namely the qubit or qudit. As a special case, it is impossible to erase azimuthal angle information of a qubit whilst keeping the polar angle information intact, which we show is the no-flipping principle. Our theorem adds new insight into the integrity of quantum information, namely that we can erase complete information but not partial information. This in turn implies that quantum information is indivisible and we have to treat quantum information as a ‘whole entity’. We also introduce the no partial erasure propositions for $SU(2)$ coherent states and for continuous variable quantum information. Since the first e-print release of our work, a no-splitting theorem for quantum information [12] has been presented, which we show is a straightforward corollary to our Theorem 4.

An arbitrary qubit is expressed as

$$|\Omega\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle \in \mathcal{H}^2 \quad (1)$$

with $\Omega \equiv (\theta, \phi)$ and

$$\theta \in [0, \pi], \phi \in [0, 2\pi). \quad (2)$$

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Each pure state is uniquely identified with a point on the Poincaré sphere with θ the polar angle and ϕ the azimuthal angle. The states $|0\rangle$ and $|1\rangle$ are the logical zero and one states, respectively. Complete erasure would map all arbitrary qubit states into a fixed qubit state $|\Omega_0\rangle = |\Sigma\rangle$ regardless of the input state parameters θ and ϕ , which is known to be impossible by unitary means. More generally, the d -dimensional analogue of the 2-dimensional qubit is a qudit with quantum state

$$|\vec{\Omega}\rangle = \sum_{i=1}^d e^{i\phi_i} \cos \frac{\theta_i}{2} |i\rangle \in \mathcal{H}^d, \quad (3)$$

with

$$\vec{\Omega} \equiv (\vec{\theta}, \vec{\phi}), \quad (4)$$

and $\cos \frac{\theta_i}{2} = |\langle i | \vec{\Omega} \rangle|$ with θ_i is the Bargmann angle between the i^{th} orthonormal vector and the qudit state.

Each vector $\vec{\theta}$ and $\vec{\phi}$ is d -dimensional, but normalization of the qudit state and the unphysical nature of the overall phase reduces the number of free parameters for the qudit to $2(d-1)$. A pure qudit can be represented as a point on the projective Hilbert space \mathcal{P} which is a real $2(d-1)$ -dimensional manifold. For the qubit case, $d=2$, and there are two parameters, so we see the reduction of the formula for two qubits is correct.

The organization of our paper is as follows. In the section II, we present our no-partial erasure theorem for non-orthogonal qubits and qudits. Also we show how the no-flipping theorem for arbitrary qubits emerges as a corollary to our theorem. Then, we prove the no-partial erasure theorem for an arbitrary qudit using linearity of quantum theory. Furthermore, we show that the no-splitting theorem for quantum information also follows from our theorem. In the section III, we present the no-partial erasure result for spin coherent state. In the section IV, we generalize the no-partial erasure theorem for continuous variable quantum information. Lastly, in the section V, we conclude our paper.

II. NO-PARTIAL ERASURE OF QUBIT AND QUDIT

In this section, we prove powerful theorems that establish the impossibility of partial erasure of arbitrary qudit states but first begin with a definition of partial erasure.

Definition 1. Partial erasure is a completely positive (CP), trace preserving mapping of all *pure* states

$$|\{\zeta_i; i = 1, \dots, n\}\rangle, \quad (5)$$

with real parameters ζ_i , in an n -dimensional Hilbert space to *pure* states in an m -dimensional Hilbert subspace via a constraint

$$\kappa(\{\zeta_i; i = 1, \dots, n\}) \quad (6)$$

such that $m < n$.

The process of partial erasure reduces the dimension of the parameter domain and *does not leave the state entangled with any other system*. One may wonder why we emphasize that the process does not entangle with other system; it is because we want to analyze this process in parallel with the complete erasure process.

Recall that, in complete erasure, an arbitrary pure state of a qubit is mapped to a fixed pure state, i.e., $|\Omega\rangle \mapsto |\Sigma\rangle = |0\rangle$. If we allow the original system to be entangled with ancilla, then we would trivially be able to erase partial information. For example, if we commence with a qubit in the state

$$|\Omega\rangle = \alpha|0\rangle + \beta|1\rangle \quad (7)$$

and enjoin an ancilla in the state $|0\rangle$, then a controlled-NOT gate would entangle these two qubits together. Then the resulting state of the original qubit has no phase information about the input state. So we have a process that maps

$$|\Omega\rangle\langle\Omega| \mapsto \rho(\theta). \quad (8)$$

Therefore, we do not want that final state of the quantum system is in a mixed state. We would like to see if the partial information can be erased and yet we retain purity of a quantum state in question. One example of partial erasure would be reducing the parameter space for the qubit from Ω to θ by fixing ϕ as a constant (say $\phi = 0$), i.e.,

$$|\theta, \phi\rangle \mapsto |\theta\rangle \quad (9)$$

corresponds to partial erasure of a qubit where the phase information or azimuthal angle information about a qubit is lost.

We now prove that there cannot exist a physical operation capable of erasing any pair of non-orthogonal qudit states.

Theorem 1. *In general, there is no physical operation that can partially erase any pair of non-orthogonal qudits.*

Proof. The partial erasure quantum operation is a CP, trace-preserving mapping that transforms an arbitrary qudit state $|\vec{\Omega}\rangle$ into the constrained qudit state $|\vec{\Omega}\rangle_\kappa$ for $\kappa(\vec{\Omega}) = 0$ a constraining equation that effectively reduces the parameter space by at least one dimension. Arbitrary qudit states can be represented as points on the projective Hilbert space parametrized by Ω , and κ constrains these points to a one-dimensional subset of the projective Hilbert space.

We can introduce the parametrization $\vec{\tau}$ so that the constraint κ allows us to write the parameters as $\vec{\Omega}(\vec{\tau})$. Consider the mapping of two distinct qudit states $|\vec{\Omega}\rangle$ and $|\vec{\Omega}'\rangle$ to $|\Omega(\vec{\tau})\rangle$ and $|\vec{\Omega}'(\vec{\tau}')\rangle$, respectively, for some values $\vec{\tau}$ and $\vec{\tau}'$.

By attaching an ancilla, the quantum operation \mathcal{E} that maps

$$|\vec{\Omega}\rangle\langle\vec{\Omega}| \mapsto \mathcal{E}(|\vec{\Omega}\rangle\langle\vec{\Omega}|) = |\vec{\Omega}(\vec{\tau})\rangle\langle\vec{\Omega}(\vec{\tau})| \quad (10)$$

can be represented as a unitary evolution on the enlarged Hilbert space, so partial erasure of qudits can be expressed as

$$\begin{aligned} |\vec{\Omega}\rangle|A\rangle &\mapsto |\vec{\Omega}(\vec{\tau})\rangle|A_{\Omega}\rangle, \\ |\vec{\Omega}'\rangle|A\rangle &\mapsto |\vec{\Omega}'(\vec{\tau}')\rangle|A_{\Omega'}\rangle, \end{aligned} \quad (11)$$

where $|A\rangle$ is the initial state, $|A_{\Omega}\rangle$, and $|A_{\Omega'}\rangle$ are the final states of the ancilla. Now, by unitarity, taking the inner product we have

$$\langle\vec{\Omega}|\vec{\Omega}'\rangle = \langle\vec{\Omega}(\vec{\tau})|\vec{\Omega}'(\vec{\tau}')\rangle\langle A_{\Omega}|A_{\Omega'}\rangle. \quad (12)$$

However, the inner product of the resultant two qudit states is not same as the inner product of the original qudit states. Hence we cannot partially erase a pair of non-orthogonal qudits by any physical operation. \square

If $d = 2$, we readily obtain the no partial erasure theorem for any pair of non-orthogonal qubits.

Example 1. As a special, and instructive, case, let us consider the impossibility of erasing azimuthal angle information for qubits. Consider the partial erasure of two non-orthogonal qubit states $|\Omega\rangle = |\theta, \phi\rangle$ and $|\Omega'\rangle = |\theta', \phi'\rangle$ by removing the azimuthal angle information. In the enlarged Hilbert space the unitary transformations for these two states are given by

$$\begin{aligned} |\theta, \phi\rangle|A\rangle &\mapsto |\theta\rangle|A_{\Omega}\rangle, \\ |\theta', \phi'\rangle|A\rangle &\mapsto |\theta'\rangle|A_{\Omega'}\rangle. \end{aligned} \quad (13)$$

As unitary evolution must preserve the inner product, we have

$$\langle\theta, \phi|\theta', \phi'\rangle = \langle\theta|\theta'\rangle\langle A_{\Omega}|A_{\Omega'}\rangle. \quad (14)$$

More explicitly, in terms of these real parameters we have

$$\cos\frac{\theta}{2}\cos\frac{\theta'}{2} + \sin\frac{\theta}{2}\cos\frac{\theta'}{2}e^{i(\phi'-\phi)} = \cos\frac{\theta-\theta'}{2}\langle A_{\Omega}|A_{\Omega'}\rangle. \quad (15)$$

However, for arbitrary values of ϕ and ϕ' the above equation cannot hold. Therefore, it is impossible to erase azimuthal angle information of a qubit by physical operations.

The above equation suggests that there may be special classes of qubit states that can be partially erased. The general condition is that if

$$\phi = \phi' + 2n\pi, \quad (16)$$

n being an integer, then any qubit that differs in phase by $2n\pi$ can be partially erased. This implies if we restrict our qubits to be chosen from any great circle passing through north and south poles of the Poincaré sphere, then those qubits can be partially erased by a physical operation. Similarly, we can show that it is impossible

to erase the information about the polar angle θ of an arbitrary qubit, i.e., the transformation

$$|\theta, \phi\rangle|A\rangle \mapsto |\phi\rangle|A_{\Omega}\rangle \quad (17)$$

is not allowed.

Remark. Although there does not exist a completely positive, trace-preserving mapping that partially erases a qubit, there exists a proper subset of qudit or qubit states that are erased by a given mapping. For example the set of qubit states whose parameters satisfy the constraint κ can have partial erasure according to the already-imposed constraint κ . Partial erasure can also be effected on an arbitrary qubit by a unitary mapping if the state is known simply because there always exists a unitary map between any two states in a Hilbert space; hence there exists a unitary mapping from every qubit state to constrained qubit states. Also a qubit or a qudit in known orthogonal states can be partially erased.

Now we show that for $d = 2$ and

$$\kappa(\Omega = (\theta, \phi)) = \kappa(\theta, \phi_0) \quad (18)$$

for all Ω , with the azimuthal phase ϕ_0 fixed, we obtain the no flipping principle for an arbitrary qubit. We know that a classical bit like 0 or 1 can be flipped, so also a qubit in an orthogonal state like $|0\rangle$ or $|1\rangle$. However, an unknown qubit $|\Omega\rangle$ cannot be flipped. That is there is no exact universal NOT gate for an arbitrary qubit. This is because the flipping operation is an anti-unitary operation which is not a CP map and thus cannot be implemented physically. The no-flipping principle for an unknown qubit is another important limitation in quantum information.

Corollary 2. *Erasure of the azimuthal phase from the parameter domain of a qubit, whilst leaving the polar phase parameter unchanged by the mapping, implies the existence of a universal NOT gate.*

Proof. Suppose we can erase phase information of an arbitrary qubit. For an orthogonal qubit state $|\theta, \phi\rangle^{\perp}$ partial erasure effects the mapping

$$|\theta, \phi\rangle^{\perp}|A\rangle \mapsto |\theta\rangle^{\perp}|A_{\Omega}\rangle^{\perp}. \quad (19)$$

If this holds, then after the partial erasure one can apply a local unitary NOT gate to $|\theta\rangle^{\perp}$ and convert it to $|\theta\rangle$ (in this case by applying $i\sigma_y$).

Next an application of the inverse of the partial erasure transformation yields the state $|\theta, \phi\rangle$. This means by applying a sequence of unitary transformations one can flip an unknown qubit state, that is, map an arbitrary qubit to its complement. Hence erasure of azimuthal phase but not polar phase implies the existence of a universal NOT gate. \square

Now we can apply the above to prove easily the non-existence of a universal NOT gate [8].

Corollary 3. *A universal NOT gate is impossible.*

Proof. To prove the no flipping principle, we show that a universal NOT gate requires partial erasure. Suppose there is a universal NOT gate for an arbitrary qubit that takes

$$|\theta, \phi\rangle \mapsto |\theta, \phi\rangle^\perp. \quad (20)$$

However, it is known that such an operation exists [13, 14] if and only if the qubit belongs to a great circle, that is, the qubit parameter domain is constrained by κ to a great circle on the sphere defined by θ and ϕ . This means that the arbitrary qubit must have been mapped to a qubit on the great circle (this mapping is a partial erasure machine) before passing through the universal NOT.

After the universal NOT it must have passed through a reverse partial erasure machine. Thus to be able to design a universal NOT gate for an arbitrary qubit we need a partial erasure operation from

$$|\theta, \phi\rangle \mapsto |\theta\rangle. \quad (21)$$

However, we know that this is impossible. Hence no partial erasure of phase information implies the non-existence of a universal NOT gate for a qubit. \square

Remark. Theorem 1 establishes that there is no physical operation that can partially erase any pair of nonorthogonal qudit states, from which the ‘no flipping principle’ emerges as a simple corollary. However, Theorem 1 applies to a set of quantum states which are not arbitrary. One can ask a more general question: Can we partially erase an arbitrary quantum state by a linear transformation? Now we show that linearity of quantum theory establishes that there cannot exist a physical operation that can partially erase a qudit, which is a stronger result.

Theorem 4. *An arbitrary qudit cannot be partially erased by an irreversible operation.*

Proof. We know that partial erasure operation for known orthogonal states is possible. Let $|\vec{\Omega}_n\rangle$ be a known orthonormal basis in \mathcal{H}^d . Then a partial erasure operation for these states yields

$$|\vec{\Omega}_n\rangle|A\rangle \mapsto |\vec{\Omega}_n(\vec{\tau})\rangle|A_{\Omega_n}\rangle. \quad (22)$$

Consider an arbitrary qudit $|\vec{\Omega}\rangle$ of Eq. (3) which is a linear superposition of the basis states $\{|\vec{\Omega}_n\rangle\}$. Suppose partial erasure of $|\vec{\Omega}\rangle$ is possible. Then linearity of the partial erasure transformation requires that

$$\begin{aligned} |\vec{\Omega}\rangle|A\rangle &= \sum_{n=1}^d e^{i\phi_n} \cos \frac{\theta_n}{2} |\vec{\Omega}_n\rangle|A\rangle \mapsto \\ &\sum_{n=1}^d e^{i\phi_n} \cos \frac{\theta_n}{2} |\vec{\Omega}_n(\vec{\tau})\rangle|A_{\Omega_n}\rangle = |\vec{\Omega}\rangle. \end{aligned} \quad (23)$$

Ideally the partial erasure of an arbitrary qudit should have yielded a *pure* output state that takes constrained values for θ_n and ϕ_n . However, the resultant state is not a pure qudit state but rather is entangled with the ancilla. By definition partial erasure maps a pure state to a pure state, hence a contradiction. Thus, linearity (including irreversible operations) prohibits partial erasure of arbitrary quantum information. \square

For $d = 2$ we obtain the impossibility of partial erasure of an arbitrary qubit. For example, we cannot omit either polar or azimuthal angle information of a qubit by irreversible operation.

Here we give another proof for no-partial erasure of an arbitrary qubit solely using linearity and without using ancilla states. Suppose we have the partial erasure operation for two known orthogonal states such as $|\Psi(\theta_0, \phi_0)\rangle$ and $|\bar{\Psi}(\theta_0, \phi_0)\rangle$. Then the partial erasure operation can be represented by

$$\begin{aligned} |\Psi(\theta_0, \phi_0)\rangle &\mapsto |\psi(\theta_0)\rangle, \\ |\bar{\Psi}(\theta_0, \phi_0)\rangle &\mapsto |\bar{\psi}(\theta_0)\rangle. \end{aligned} \quad (24)$$

Let an arbitrary qubit $|\Phi(\theta, \phi)\rangle$ be in a linear superposition of these two basis states:

$$|\Phi(\theta, \phi)\rangle = \cos \frac{\theta}{2} |\Psi(\theta_0, \phi_0)\rangle + \sin \frac{\theta}{2} \exp(i\phi) |\bar{\Psi}(\theta_0, \phi_0)\rangle. \quad (25)$$

If we want to have partial erasure of $|\Phi(\theta, \phi)\rangle$ then, by linearity of the erasure transformation we have

$$\begin{aligned} |\Phi(\theta, \phi)\rangle &= \cos \frac{\theta}{2} |\Psi(\theta_0, \phi_0)\rangle + \sin \frac{\theta}{2} e^{i\phi} |\bar{\Psi}(\theta_0, \phi_0)\rangle \\ &\mapsto \cos \frac{\theta}{2} |\psi(\theta_0)\rangle + \sin \frac{\theta}{2} e^{i\phi} |\bar{\psi}(\theta_0)\rangle \\ &= |\tilde{\Phi}(\theta, \phi)\rangle \end{aligned} \quad (26)$$

Again, ideally the partial erasure of an arbitrary qubit should have yielded an output state that is completely independent of ϕ , i.e., $|\Phi(\theta, \phi)\rangle \mapsto |\Phi(\theta)\rangle$. However, we have a state $|\tilde{\Phi}(\theta, \phi)\rangle$ that has complete information about both θ and ϕ . Hence, this shows that linearity (which includes also irreversible operations) of quantum theory does not allow partial erasure of quantum information. If we include ancilla, then the original qubit will be entangled with the ancilla and by throwing out ancilla, we will be left with a qubit state that is no more pure. Note that if we allow irreversible operation (unitary evolution of combined system and tracing out of the ancilla), we can eliminate complete information of an arbitrary qubit (albeit the fact that the original information still remains in the ancilla) as in the complete erasure. Thus, one can erase the complete information of a qubit but not the partial information by an irreversible operation and yet retain its purity.

The implication of being able to completely erase, but not partially erase quantum information implies that quantum information is indivisible. There is no classical

analogue for this result: no partial erasure is a strictly quantum phenomenon. We introduce the term *integrity principle* to refer to this inability to partially erase quantum information.

Since the release of our e-print proving ‘no partial erasure’ theorem a ‘no-splitting theorem’ for quantum information has been presented [12], where ‘no splitting’ refers to the impossibility of splitting a qubit $|\theta, \phi\rangle$ into a product state $|\theta\rangle|\phi\rangle$ with one qubit representing the θ information and the other representing the ϕ information. Here we show that the no-splitting follows from Theorem 4.

Corollary 5. *No-partial erasure theorem implies a no-splitting of quantum information.*

Proof. Suppose quantum information can be split. Then there exists an operation that transforms

$$|\theta, \phi\rangle \mapsto |\theta\rangle|\phi\rangle. \quad (27)$$

We can append an ancillary qubit in a specific state and swap with the second qubit, then trace to eliminate all information about ϕ . Thus splitting implies partial erasure, which contradicts Theorem 4. Hence, it is impossible to split quantum information. \square

III. NO-PARTIAL ERASURE OF SPIN COHERENT STATE

Our theorem that no partial erasure of qudits is possible is important because quantum information is clearly not only conserved but also indivisible. However, the ‘no partial erasure’ theorem yields another important result for erasure of spin coherent states, also known as $SU(2)$ coherent states [15, 16, 17].

The $SU(2)$ coherent states are a generalization of qubits, which can be thought of as spin- $\frac{1}{2}$ states, to states of higher spin j . The $SU(2)$ raising and lowering operators are \hat{J}_+ and \hat{J}_- , respectively, and their commutator

$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z \quad (28)$$

yields the weight operator \hat{J}_z with spectrum

$$\{m; -j \leq m \leq j\}$$

and integer spacing between successive values of m . The weight basis is $|j m\rangle$ with $j(j+1)$ the eigenvalue for states in the j^{th} irrep of the Casimir invariant \hat{J}^2 .

The $SU(2)$ coherent states are obtained by ‘rotations’ of the highest-weight state $|j j\rangle$. Here we use the stereographic parameter

$$\gamma = e^{i\phi} \tan(\theta/2) \quad (29)$$

that corresponds to the coordinates of the state on the complex plane obtained by a stereographic projection of

the point on the Poincaré sphere for the given state, with parameters θ and ϕ are the polar and azimuthal angular coordinates of the Poincaré sphere defined earlier; here the sphere represents states of a $(2j+1)$ -dimensional system, not just the two-dimensional qubit. For $|j j\rangle$ the highest-weight state, the $SU(2)$ coherent state is [18]

$$|j, \gamma\rangle = R_j(\gamma)|j j\rangle = \sum_{m=0}^{2j} \binom{2j}{m}^{1/2} \frac{\gamma^m}{(1+|\gamma|^2)^j} |j j-m\rangle \quad (30)$$

for

$$\begin{aligned} R_j(\gamma) &= \exp\left[\frac{1}{2}\theta\left(\hat{J}_-e^{i\phi} - \hat{J}_+e^{-i\phi}\right)\right] \\ &= \exp(\gamma\hat{J}_-) \exp[-\hat{J}_z \ln(1+|\gamma|^2)] \exp(-\gamma^*\hat{J}_+). \end{aligned} \quad (31)$$

We can now prove the following no go result using our theorem.

Corollary 6. *Partial erasure of $SU(2)$ coherent states is impossible.*

Proof. The $SU(2)$ coherent state is a qudit with the constraint that

$$e^{i\phi_m} \cos \frac{\theta_m}{2} = \binom{2j}{m}^{1/2} \frac{\gamma^m}{(1+|\gamma|^2)^j} \quad (32)$$

for each m . Partial erasure of the $SU(2)$ coherent states corresponds to partial erasure over a subspace of qudits, which we have shown is impossible. \square

IV. NO-PARTIAL ERASURE OF CONTINUOUS VARIABLE STATE

Next, we prove the ‘no partial erasure’ theorem for continuous variable quantum information. Ideally continuous variable (CV) quantum information encodes quantum information as superpositions of eigenstates of the position operator \hat{x} , namely

$$\hat{x}|x\rangle = x|x\rangle; \{x \in \mathbb{R}\} \quad (33)$$

with complex amplitude $\Psi(x)$ [19]. We can represent a CV state as

$$|\Psi\rangle = \int_{\mathbb{R}} dx \Psi(x)|x\rangle, \quad \Psi(x) = \langle x|\Psi\rangle. \quad (34)$$

Note that $\Psi(x)$ can be any complex-valued function, subject to the requirement of square-integrability and normalization.

Now let us reduce $\Psi(x)$ to a real-valued function, so we have effectively reduced the parameter space even in infinity dimensional Hilbert space. Does there exist a completely positive, trace preserving mapping from the set of states with $\Psi(x)$ a general complex-valued function to the new $\psi(x)$ a general real-valued function?

Definition 2. Partial erasure of continuous variable quantum information is a completely positive map of all arbitrary pure states with complex wavefunctions to pure states with real wavefunctions.

Theorem 7. *There is no physical operation that can partially erase any pair of complex wavefunctions.*

Proof. We prove this theorem for a system with one degree of freedom, namely canonical position x ; this proof is readily extended to more than one degrees of freedom. Suppose there is a CP map that can partially erase a wavefunction $\Psi(x)$ via

$$|\Psi\rangle|A\rangle \mapsto |\psi\rangle|A_\Psi\rangle, \quad (35)$$

where

$$\|\Psi\|^2 = \int_{\mathbb{R}} dx |\Psi(x)|^2, \quad \|\psi\|^2 = \int_{\mathbb{R}} dx \psi(x)^2. \quad (36)$$

If this holds for another arbitrary wavefunction $\Phi(x)$, then we have

$$|\Phi\rangle|A\rangle \mapsto |\phi\rangle|A_\Phi\rangle, \quad (37)$$

where

$$\|\Phi\|^2 = \int_{\mathbb{R}} dx |\Phi(x)|^2 \quad (38)$$

and

$$\|\phi\|^2 = \int_{\mathbb{R}} dx \phi(x)^2. \quad (39)$$

However, the inner product preserving condition

$$\int_{\mathbb{R}} dx \Psi(x)^* \Phi(x) = \int_{\mathbb{R}} dx \psi(x) \phi(x) \int_{\mathbb{R}} dx A_\Psi^*(x) A_\Phi(x) \quad (40)$$

cannot hold for general complex-valued wavefunctions. Hence, we cannot partially erase a pair of complex wavefunction. \square

This result is analogous to partial erasure of qudits. Furthermore, the restriction should apply for any erasure of the complex domain by one dimension (such as a circle where amplitude is fixed and phase varies). Similarly, one can give a general proof of no partial erasure of continuous variable quantum information, not just complex to real but complex to any one-dimensional subset of the complex space.

For example, let the partial erasure process transforms the wavefunction such that one of the complex amplitudes becomes real (which is one way to reduce the parameter space by one dimension). Then we can prove that it is also impossible. Under partial erasure the continuous variable state

$$|\Psi\rangle = \int_{\mathbb{R}} dx \Psi(x)|x\rangle, \quad (41)$$

with

$$\Psi(x) = \sum_{n=0}^{\infty} c_n \Psi_n(x), \quad (42)$$

$c_n = |c_n| \exp(i\theta_n)$ transforms as

$$\sum_{n=0}^{\infty} c_n \Psi_n(x) \mapsto \sum_{n=0}^{\infty} d_n \Psi_n(x) \quad (43)$$

where the constraint is that all $c_n = d_n$ are complex except for one d_k , which is a real number. Consider a pair of wavefunctions $(\Psi(x), \Phi(x))$ with

$$\Phi(x) = \sum_{n=0}^{\infty} c'_n \Phi_n(x), \quad (44)$$

$c'_n = |c'_n| \exp(i\theta'_n)$ and partial erasure of $\Phi(x)$ is given by

$$\sum_{n=0}^{\infty} c'_n \Phi_n(x) \mapsto \sum_{n=0}^{\infty} d'_n \Phi_n(x) \quad (45)$$

with similar constraints.

For clarity, let us not include an ancilla. Unitarity implies that

$$\exp(i[\theta'_k - \theta_k]) = 1, \quad (46)$$

which is impossible for arbitrary values of θ_k and θ'_k . Hence, we cannot forget even one parameter of the complex wavefunction.

V. CONCLUSIONS

In summary we have introduced a new process called partial erasure of quantum information and asked if quantum information can undergo partial erasure. We have shown that partial erasure of qubits, qudits, $SU(2)$ coherent states, and continuous variable quantum information is impossible. These results point to the integrity principle for quantum information, namely that it is indivisible and robust even against partial erasure. This principle gives a new meaning to quantum information and nicely complements the recent profound observation of the principle of conservation of quantum information [11].

Furthermore, the impossibility theorems presented here underscore essential differences between classical information (which could be stored in orthogonal quantum states) and general quantum information, analogous to related but distinct impossibility results such as the no-cloning, no-deleting and no-flipping principles. Our principle of quantum information integrity may have implications for investigations into quantum mechanics over real, complex, and quaternionic number fields [20, 21]: a unitary equivalence between complex and real quantum

theories would appear to contradict the no partial erasure theorem. Interesting problems that warrants further investigation is approximate deterministic partial erasure and exact probabilistic partial erasure over restricted classes of states.

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APPENDIX A: NO COMPLETE ERASURE BY REVERSIBLE OPERATIONS

In quantum theory a reversible operation can be represented by a unitary operator. Erasure of a qubit state $|\Psi\rangle$ transforms this to a fixed state $|\Sigma\rangle$, which contains no information about the input state.

Consider erasure of a pair of qubits $|\Omega\rangle$ and $|\Omega'\rangle$ such that

$$|\Omega\rangle \mapsto |\Sigma\rangle \quad (\text{A1})$$

and

$$|\Omega'\rangle \mapsto |\Sigma\rangle. \quad (\text{A2})$$

As unitary evolution preserves the inner product, we will have

$$\langle\Omega|\Omega'\rangle = 1, \quad (\text{A3})$$

which cannot be true. Furthermore, even for two orthogonal states such as $|0\rangle$ and $|1\rangle$, this evolution implies a contradiction.

This paradox demonstrates, in a simplest and yet profound way, that neither classical information nor quantum information can be erased by any reversible operation.

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